OPERATIONS RESEARCH

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TRANSPORTATION MODELS

The transportation problem is a special type of linear programming problem, where the objective is to minimize the cost of distributing a product from a number of sources to a number of destinations. The transportation problem deals with a special class of linear programming problems in which the objective is to transport a homogeneous product manufactured at several plants (origins) to a number of different destinations at a minimum total cost. The total supply available at the origin and the total quantity demanded by the destinations are given in the statement of the problem. The cost of shipping a unit of goods from a known origin to a known destination is also given. The objective is to determine the optimal allocation that results in minimum total shipping cost.

Mathematical Representation of Transportation Problem

A firm has 3 factories - A, E, and K. There are four major warehouses situated at B, C, D, and M. Average daily product at A, E, K is 30, 40, and 50 units respectively. The average daily requirement of this product at B, C, D, and M is 35, 28, 32, 25 units respectively. The transportation cost (in Rs.) per unit of product from each factory to each warehouse is given below:

Warehouse					
Factory	B	C	D	M	Supply
A	6	8	8	5	30
E	5	11	9	7	40
K	8	9	7	13	50
Demand	35	28	32	25	

The problem is to determine a routing plan that minimizes total transportation costs.

Let x_{ij} = no. of units of a product transported from ith factory($i = 1, 2, 3$) to jth warehouse

($j = 1, 2, 3, 4$).

It should be noted that if in a particular solution the x_{ij} value is missing for a cell, this means that nothing is shipped between factory and warehouse.

The problem can be formulated mathematically in the linear programming form as
Minimize = $6x_{11} + 8x_{12} + 8x_{13} + 5x_{14}$
$+ 5x_{21} + 11x_{22} + 9x_{23} + 7x_{24}$
$+ 8x_{31} + 9x_{32} + 7x_{33} + 13x_{34}$

subject to
Capacity constraints
$x_{11} + x_{12} + x_{13} + x_{14} = 30$
$x_{21} + x_{22} + x_{23} + x_{24} = 40$
$x_{31} + x_{32} + x_{33} + x_{34} = 50$

Requirement constraints
$x_{11} + x_{21} + x_{31} = 35$
$x_{12} + x_{22} + x_{32} = 28$
$x_{13} + x_{23} + x_{33} = 32$
$x_{14} + x_{24} + x_{34} = 25$

$x_{ij} \geq 0$

The above problem has 7 constraints and 12 variables.Since no. of variables is very high, simplex method is not applicable. Therefore, more efficient methods have been developed to solve transportation problems.

The general mathematical model may be given as follows

minimize $c_{ij}x_{ij}$

subject to

x_{ij} □ S_i for $i = 1,2, ..., m$ (supply)
x_{ij} □ D_j for $j = 1,2, ..., n$ (demand)

x_{ij} □ 0

For a feasible solution to exist, it is necessary that total capacity equals total requirements.

Total supply = total demand.
Or $\square\, a_i = \square\, b_j$.

If total supply = total demand then it is a balanced transportation problem.
There will be (m + n – 1) basic independent variables out of (m x n) variables.

Only a single type of commodity is being shipped from an origin to a destination.

Total supply is equal to the total demand.

$$S_i = \qquad D_j$$

S_i (supply) and D_j (demand) are all positive integers.

Terms used in Transportation Problems.

1. Origin

It is the location from which shipments are dispatched.

2. Destination

It is the location to which shipments are transported.

3. Unit Transportation cost

It is the cost of transporting one unit of the consignment from an origin to a destination.

4. Perturbation Technique

It is a method used for modifying a degenerate transportation problem, so that the degeneracy can be resolved.

5.Feasible Solution

A solution that satisfies the row and column sum restrictions and also the non-negativity restrictions is a feasible solution.

6. Basic Feasible Solution

A feasible solution of (m X n) transportation problem is said to be basic feasible solution, when the total number of allocations is equal to (m + n – 1).

7. Optimal Solution

A feasible solution is said to be optimal solution when the total transportation cost will be the minimum cost.

In the sections that follow, we will concentrate on algorithms for finding solutions to transportation problems.

Methods for finding an initial basic feasible solution:

1.North West Corner Rule

2.Matrix Minimum Method

3.Vogel Approximation Method

North West Corner Rule

The North West corner rule is a method for computing a basic feasible solution of a transportation problem, where the basic variables are selected from the North – West corner (i.e., top left corner).The standard North West Corner Rule instructions are paraphrased below:

Steps

1.Select the north west (upper left-hand) corner cell of the transportation table and allocate as many units as possible equal to the minimum between available supply and demand, i.e., min(s1, d1).

2.Adjust the supply and demand numbers in the respective rows and columns.

3.If the demand for the first cell is satisfied, then move horizontally to the next cell in the second column.

4.If the supply for the first row is exhausted, then move down to the first cell in the second row.

5.If for any cell, supply equals demand, then the next allocation can be made in cell either in the next row or column.

Continue the process until all supply and demand values are exhausted.

Matrix Minimum Method

Matrix minimum (Least cost) method is a method for computing a basic feasible solution of a transportation problem, where the basic variables are chosen according to the unit cost of transportation. This method is very useful because it reduces the computation and the time required to determine the optimal solution. The following steps summarize the approach.

Steps

1. Identify the box having minimum unit transportation cost (c_{ij}).
2. If the minimum cost is not unique, then you are at liberty to choose any cell.
3. Choose the value of the corresponding x_{ij} as much as possible subject to the capacity and requirement constraints.

Repeat steps 1-3 until all restrictions are satisfied

Vogel Approximation Method

The Vogel approximation (Unit penalty) method is an iterative procedure for computing a basic feasible solution of a transportation problem. This method is preferred over the other two methods, because the initial basic feasible solution obtained by this method is either optimal or very close to the optimal solution.

Steps

The standard instructions are paraphrased below:

1.Identify the boxes having minimum and next to minimum transportation cost in each row and write the difference (penalty) along the side of the table against the corresponding row.

2.Identify the boxes having minimum and next to minimum transportation cost in each column and write the difference (penalty) against the corresponding column

3.Identify the maximum penalty. If it is along the side of the table, make maximum allotment to the box having minimum cost of transportation in that row. If it is below the table, make maximum allotment to the box having minimum cost of transportation in that column.

4.If the penalties corresponding to two or more rows or columns are equal, you are at liberty to break the tie arbitrarily.

Repeat the above steps until all restrictions are satisfied.

Optimal / Final Solution

After computing an initial basic feasible solution, we must now proceed to determine whether the solution so obtained is optimal or not. In the next section, we will discuss about the methods used for finding an optimal solution.

1.Stepping Stone Method

It is a method for finding the optimum solution of a transportation problem.

Steps

1. Determine an initial basic feasible solution using any one of the following:

- North West Corner Rule
- Matrix Minimum Method
- Vogel Approximation Method

2. Make sure that the number of occupied cells is exactly equal to m+n-1, where m is the number of rows and n is the number of columns.

3. Select an unoccupied cell. Beginning at this cell, trace a closed path, starting from the selected unoccupied cell until finally returning to that same unoccupied cell.

4. Assign plus (+) and minus (-) signs alternatively on each corner cell of the closed path just traced, beginning with the plus sign at unoccupied cell to be evaluated.

5. Add the unit transportation costs associated with each of the cell traced in the closed path. This will give net change in terms of cost.

6. Repeat steps 3 to 5 until all unoccupied cells are evaluated.

7. Check the sign of each of the net change in the unit transportation costs. If all the net changes computed are greater than or equal to zero, an optimal solution has been reached. If not, it is possible to improve the current solution and decrease the total transportation cost, so move to step 8..

8. Select the unoccupied cell having the most negative net cost change and determine the maximum number of units that can be assigned to this cell. The smallest value with a negative position on the closed path indicates the number of units that can be shipped to the entering cell. Add this number to the unoccupied cell and to all other cells on the path marked with a plus sign. Subtract this number from cells on the closed path marked with a minus sign.

2.Modified Distribution Method (MODI) or (u - v) method

The modified distribution method, also known as MODI method or (u - v) method provides a minimum cost solution to the transportation problem. In the stepping stone method, we have to draw as many closed paths as equal to the unoccupied cells for their evaluation. To the contrary, in MODI method, only closed path for the unoccupied cell with highest opportunity cost is drawn.

Steps

1. Determine an initial basic feasible solution using any one of the three methods given below:

- North West Corner Rule
- Matrix Minimum Method
- Vogel Approximation Method

2. Determine the values of dual variables, u_i and v_j, using $u_i + v_j = c_{ij}$

3. Compute the opportunity cost using $c_{ij} – (u_i + v_j)$.

4. Check the sign of each opportunity cost. If the opportunity costs of all the unoccupied cells are either positive or zero, the given solution is the optimal solution. On the other hand, if one or more unoccupied cell has negative opportunity cost, the given solution is not an optimal solution and further savings in transportation cost are possible.

5. Select the unoccupied cell with the smallest negative opportunity cost as the cell to be included in the next solution.

6. Draw a closed path or loop for the unoccupied cell selected in the previous step. Please note that the right angle turn in this path is permitted only at occupied cells and at the original unoccupied cell.

7. Assign alternate plus and minus signs at the unoccupied cells on the corner points of the closed path with a plus sign at the cell being evaluated.

8. Determine the maximum number of units that should be shipped to this unoccupied cell. The smallest value with a negative position on the closed path indicates the number of units that can be shipped to the entering cell. Now, **add** this quantity to all the cells on the corner points of the closed path marked with plus signs, and **subtract** it from those cells marked with minus signs. In this way, an unoccupied cell becomes an occupied cell.

9. Repeat the whole procedure until an optimal solution is obtained.

Unbalanced Transportation Problem

So far we have assumed that the total supply at the origins is equal to the total requirement at the destinations.

Specifically, Si = Dj

But in certain situations, the total supply is not equal to the total demand. Thus, the transportation problem with unequal supply and demand is said to be unbalanced transportation problem.

If the total supply is more than the total demand, we introduce an additional column, which will indicate the surplus supply with transportation cost zero. Similarly, if the total demand is more than the total supply, an additional row is introduced in the table, which represents unsatisfied demand with transportation cost zero. The balancing of an unbalanced transportation problem is illustrated in the following example.

Example

Plant	Warehouse			Supply
	W1	W2	W3	
A	28	17	26	500
B	19	12	16	300
Demand	250	250	500	

Solution:

The total demand is 1000, whereas the total supply is 800.

Si < Dj

Total supply < total demand.

To solve the problem, we introduce an additional row with transportation cost zero indicating the unsatisfied demand.

Plant	Warehouse			Supply
	W1	W2	W3	
A	28	17	26	500
B	19	12	16	300
Unsatisfied demand	0	0	0	200

Demand	250	250	500	1000

Using matrix minimum method, we get the following allocations.

Plant	Warehouse			Supply
	W1	W2	W3	
A		17		500
B	19			300
Unsatisfied demand		0	0	200
Demand	250	250	500	1000

Initial basic feasible solution

50 X 28 + 450 X 26 + 250 X 12 + 50 X 16 + 200 X 0 = 16900

Maximization in A Transportation Problem

There are certain types of transportation problems where the objective function is to be maximized instead of being minimized. These problems can be solved by converting the maximization problem into a minimization problem.

Example

Surya Roshni Ltd. has three factories - X, Y, and Z. It supplies goods to four dealers spread all over the country. The production capacities of these factories are 200, 500 and 300 per month respectively.

Factory	Dealer				Capacity
	A	B	C	D	
X	12	18	6	25	200
Y	8	7	10	18	500
Z	14	3	11	20	300
Demand	180	320	100	400	

Determine a suitable allocation to maximize the total net return.

Solution.

Maximization transportation problem can be converted into minimization transportation problem by subtracting each transportation cost from maximum transportation cost.

Here, the maximum transportation cost is 25. So subtract each value from 25. The revised transportation problem is shown below.

Table 1

Factory	Dealer				Capacity
	A	B	C	D	
X	13	7	19	0	200
Y	17	18	15	7	500
Z	11	22	14	5	300
Demand	180	320	100	400	

An initial basic feasible solution is obtained by matrix-minimum method and is shown in the final table.

Final table

Factory	Dealer				Capacity
	A	B	C	D	
X	13	7	19		200
Y				7	500
Z		22	14		300
Demand	180	320	100	400	

The maximum net return is

25 X 200 + 8 X 80 + 7 X 320 + 10 X 100 + 14 X 100 + 20 X 200 = 14280.

Prohibited Routes

Sometimes there may be situations, where it is not possible to use certain routes in a transportation problem. For example, road construction, bad road conditions, strike, unexpected floods, local traffic rules, etc. We can handle such type of problems in different ways:

A very large cost represented by M or is assigned to each of such routes, which are not available.

To block the allocation to a cell with a prohibited route, we can cross out that cell.

The problem can then be solved in its usual way.

Example

Consider the following transportation problem.

Factory	Warehouse			Supply
	W1	W2	W3	
F1	16		12	200
F2	14	8	18	160
F3	26		16	90
Demand	180	120	150	450

Solution.

An initial solution is obtained by the matrix minimum method and is shown in the final table.

Final Table

Factory	Warehouse			Supply
	W1	W2	W3	
F1				200
F2			18	160

F3			16	90
Demand	180	120	150	450

Initial basic feasible solution

16 X 50 + 12 X 150 + 14 X 40 + 8 X 120 + 26 X 90 = 6460.

The minimum transportation cost is Rs. 6460.

Time Minimizing Problem

Succinctly, it is a transportation problem in which the objective is to minimize the time. This problem is same as the transportation problem of minimizing the cost, expect that the unit transportation cost is replaced by the time tij.

Steps

1. Determine an initial basic feasible solution using any one of the following:

North West Corner Rule

Matrix Minimum Method

Vogel Approximation Method

2. Find Tk for this feasible plan and cross out all the unoccupied cells for which tij □ Tk.

3. Trace a closed path for the occupied cells corresponding to Tk. If no such closed path can be formed, the solution obtained is optimum otherwise, go to step 2.

Example 1

The following matrix gives data concerning the transportation times tij

Destination							
Origin	D1	D2	D3	D4	D5	D6	Supply
O1	25	30	20	40	45	37	37
O2	30	25	20	30	40	20	22

O3	40	20	40	35	45	22	32
O4	25	24	50	27	30	25	14
Demand	15	20	15	25	20	10	

Solution.

We compute an initial basic feasible solution by north west corner rule which is shown in table 1

Table 1

Destination							
Origin	D1	D2	D3	D4	D5	D6	Supply
O1				40	45	37	37
O2	30	25			40	20	22
O3	40	20	40			22	32
O4	25	24	50	27			14
Demand	15	20	15	25	20	10	

Here, $t_{11} = 25$, $t_{12} = 30$, $t_{13} = 20$, $t_{23} = 20$, $t_{24} = 30$, $t_{34} = 35$, $t_{35} = 45$, $t_{45} = 30$, $t_{46} = 25$

Choose maximum from t_{ij}, i.e., $T_1 = 45$. Now, cross out all the unoccupied cells that are □ T_1.
The unoccupied cell (O3D6) enters into the basis as shown in table 2.

Choose the smallest value with a negative position on the closed path, i.e., 10. Clearly only 10 units can be shifted to the entering cell. The next feasible plan is shown in the following table.

Table 3

Destination							
Origin	D1	D2	D3	D4	D5	D6	Supply
O1				40		37	37
O2	30	25			40	20	22
O3	40	20	40				32
O4	25	24		27		25	14
Demand	15	20	15	25	20	10	

Here, T2 = Max(25, 30, 20, 20, 20, 35, 45, 22, 30) = 45. Now, cross out all the unoccupied cells that are □ T2.

By following the same procedure as explained above, we get the following revised matrix.

Table 6

Destination							
Origin	D1	D2	D3	D4	D5	D6	Supply
O1						37	37
O2	30	25				20	22
O3		20					32
O4	25	24		27		25	14
Demand	15	20	15	25	20	10	

T3 = Max(25, 30, 20, 20, 30, 40, 35, 22, 30) = 40. Now, cross out all the unoccupied cells that are □ T3.

Now we cannot form any other closed loop with T3.
Hence, the solution obtained at this stage is optimal.
Thus, all the shipments can be made within 40 units.

Trans shipment Model

In a transportation problem, consignments are always transported from an origin to a destination. But, there could be several situations where it might be economical to transport items via one or more intermediate centres (or stages). In a transshipment problem, the available commodity is not sent directly from sources to destinations, i.e., it passes through one or more intermediate points before reaching the actual destination. For instance, a company may have regional warehouses that distribute the products to smaller district warehouses, which in turn ship to the retail stores. Succinctly, the transshipment model is an extension of the classical transportation model where an item available at point i is shipped to demand point j through one or more intermediate points.

Example

A company has nine large stores located in several states. The sales department is interested in reducing the price of a certain product in order to dispose all the stock now in hand. But, before that the management wants to reposition its stock among the nine stores according to its sales expectations at each location.

The above figure shows the numbered nodes (9 stores). A positive value next to a store represents the amount of inventory to be redistributed to the rest of the system. A negative value represents the shortage of stock. Thus, stores 1 and 4 have excess stock of 10 & 2 items respectively. Stores 3, 6 & 8 need 3, 1, and 8 more items respectively. The inventory positions of stores 2, 5 & 7 are to remain unchanged.

An item may be shipped through stores 2, 4, 5, 6, 7 & 8. These locations are known as transshipment points. Each remaining store is a source if it has excess stock, and a sink if it needs stock. In the above figure, store 1 is a source and store 3 is a sink.

The value cij is the cost of transporting items. To transport an item from store 1 to store 3, the total shipping cost is c12 + c23

In the following example, you will learn how to convert a transshipment problem to a standard transportation problem.

Example

Consider a transportation problem where the origins are plants and destinations are depots. The unit transportation costs, capacity at the plants, and the requirements at the depots are indicated below:

Table 1

Plant	Depot			
	X	Y	Z	
A	1	3	15	150
B	3	5	25	300
	150	150	150	450

When each plant is also considered a destination and each depot is also considered an origin, there are altogether five origins and five destinations. Some additional cost data are also necessary. These are presented in the following Tables.

Table 2

Unit Transportation Cost from Plant to Plant		
From Plant	To	
	Plant A	Plant B
A	0	65
B	1	0

Table 3

Unit Transportation Cost from Depot to Depot			
From Depot	To		
	Depot X	Depot Y	Depot Z
X	0	23	1
Y	1	0	3
Z	65	3	0

Table 4

Unit Transportation Cost from Depot to Plant		
Depot	Plant	
	A	B
X	3	15
Y	25	3
Z	45	55

Solution.

From Table 1, Table 2, Table 3 and Table 4 we obtain the transportation formulation of the transshipment problem.

Table 5

Transshipment Table						
	A	B	X	Y	Z	Capacity
A	0	65	1	3	15	150 + 450 = 600
B	1	0	3	5	25	300 + 450 = 750

X	3	15	0	23	1	450
Y	25	3	1	0	3	450
Z	45	55	65	3	0	450
Requirement	450	450	150 + 450 =600	150 + 450 =600	150 + 450 =600	2700

The transportation model is extended and now it includes five supply points & demand points. To have a supply and demand from all the points, a fictitious supply and demand quantity (buffer stock) of 450 is added to both supply and demand of all the points. An initial basic feasible solution is obtained by the Vogel's Approximation method and is shown in the final table.

Final Table

Trans shipment Table						
	A	B	X	Y	Z	Capacity
A		65			15	600
B			3	5	25	750
X	3	15		23		450
Y	25	3	1		3	450
Z	45	55	65	3		450
Requirement	450	450	600	600	600	2700

The total transhipment cost is:
0 X 150 + 1 X 300 + 3 X 150 + 1 X 300 + 0 X 450 + 0 X 300 + 1 X 150 + 0 X 450 + 0 X 450 = 1200

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