

OPERATIONS RESEARCH

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LINEAR PROGRAMMING PROBLEMS

Every organization faces the problem of allocating limited resources to different activities. Such type of problem arises when there are alternative ways of performing a number of activities. Linear Programming (LP) is a versatile technique for assigning a fixed amount of resources among competing factors, in such a way that some objective is optimized and other defined conditions are also satisfied. In other words, linear programming is a mathematical technique for determining the optimal allocation of resources and obtaining a particular objective when there are alternative uses of the resources. The objective may be cost minimization or inversely profit maximization.

Importance of Linear Programming

Linear programming has been successfully applied to a variety of problems of management, such as production, advertising, transportation, refinery operation, investment analysis, etc. Over the years, linear programming has been found useful not only in business and industry but also in non-profit organizations such as government, hospitals, libraries, education, etc. Actually, linear programming improves the quality of decisions by amplifying the analytic abilities of a decision maker.

The word 'linear' means that the relationships are represented by straight lines, i.e., the relationships are of the form $k = p + qx$. In other words, it is used to describe the relationships among two or more variables, which are directly proportional. The word 'programming' is concerned with optimal allocation of limited resources.

Terms employed in the description of Linear Programming Models.

Linear Function

A linear function contains terms each of which is composed of only a single, continuous variable raised to (and only to) the power of 1.

Objective Function

It is a linear function of the decision variables expressing the objective of the decision-maker. The most typical forms of objective functions are: maximize $f(x)$ or minimize $f(x)$.

Decision Variables

These are economic or physical quantities whose numerical values indicate the solution of the linear programming problem. These variables are under the control of the decision-maker and could have an impact on the solution to the problem under consideration. The relationships among these variables should be linear.

Constraints

These are linear equations arising out of practical limitations. The mathematical forms of the constraints are:

$$f(x) \leq b \text{ or } f(x) \geq b \text{ or } f(x) = b$$

Feasible Solution

Any non-negative solution which satisfies all the constraints is known as a feasible solution. The region comprising all feasible solutions is referred to as feasible region.

Optimal Solution

The solution where the objective function is maximized or minimized is known as optimal solution.

General Linear Programming Problem

Consider the following

Optimize (maximize or minimize)

$$z = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n$$

subject to

Terms used in Graphical Method

Feasible Region

It is the collection of all feasible solutions. In the following figure, the shaded area represents the feasible region.

Convex Set

A region or a set R is convex, if for any two points on the set R, the segment connecting those points lies entirely in R. In other words, it is a collection of points such that for any two points on the set, the line joining the points belongs to the set. In the following figure, the line joining P and Q belongs entirely in R.

Thus, the collection of feasible solutions in a linear programming problem form a convex set.

Redundant Constraint

It is a constraint that does not affect the feasible region.

Extreme Point

Extreme points are referred to as vertices or corner points. In the following figure, P, Q, R and S are extreme points.

Steps in Graphical Method to LPP

1. Formulate the mathematical model of the given linear programming problem.
2. Treat inequalities as equalities and then draw the lines corresponding to each equation and non-negativity restrictions.
3. Locate the end points (corner points) on the feasible region.
4. Determine the value of the objective function corresponding to the end points determined in step 3.
5. Find out the optimal value of the objective function.

Multiple Optimal Solutions

The linear programming problems discussed in the previous section possessed unique solutions. This was because the optimal value occurred at one of the extreme points (corner points). But situations may arise, when the optimal solution obtained is not unique. This case may arise when the line representing the objective function is parallel to one of the lines bounding the feasible region.

Infeasible Problem

In some cases, there is no feasible solution area, i.e., there are no points that satisfy all constraints of the problem. An infeasible LP problem with two decision variables can be identified through its graph. For example, let us consider the following linear programming problem.

3. Unbounded Solutions

It is a solution whose objective function is infinite. If the feasible region is unbounded then one or more decision variables will increase indefinitely without violating feasibility, and the value of the objective function can be made arbitrarily large. Consider the following model:

Limitations of Linear Programming

1. Linearity of relations:

A primary requirement of linear programming is that the objective function and every constraint must be linear. However, in real life situations, several business and industrial problems are nonlinear in nature.

2. Single objective:

Linear programming takes into account a single objective only, i.e., profit maximization or cost minimization. However, in today's dynamic business environment, there is no single universal objective for all organizations.

3. Certainty:

Linear Programming assumes that the values of co-efficient of decision variables are known with certainty. Due to this restrictive assumption, linear programming cannot be applied to a wide variety of problems where values of the coefficients are probabilistic. Constant parameters: Parameters appearing in LP are assumed to be constant, but in practical situations it is not so.

4. Divisibility:

In linear programming, the decision variables are allowed to take non-negative integer as well as fractional values. However, we quite often face situations where the planning models contain integer valued variables. For instance, trucks in a fleet, generators in a powerhouse, pieces of equipment, investment alternatives and there are a myriad of other examples. Rounding off the solution to the nearest integer will not yield an optimal solution. In such cases, linear programming techniques cannot be used.

Simplex Models

Although the graphical method is an invaluable aid to understand the properties of linear programming models, it provides very little help in handling practical problems. But simplex method is used for solving linear programming problems with a larger number of variables.

Many different methods have been proposed to solve linear programming problems, but simplex method has proved to be the most effective. This technique will nurture the insight needed for a sound understanding of several approaches to other programming models. This method is applicable to any problem that can be formulated in terms of linear objective function, subject to a set of linear constraints. Often, this method is termed Dantzig's simplex method, in honour of the mathematician who devised the approach.

Terms used in simplex method

Slack variable

It is a variable that is added to the left-hand side of a less than or equal to type constraint to convert the constraint into an equality. In economic terms, slack variables represent

left-over or unused capacity.

Specifically:

$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \dots + a_{inx_n} \leq b_i$ can be written as

$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \dots + a_{inx_n} + s_i = b_i$

Where $i = 1, 2, \dots, m$

Surplus variable

It is a variable subtracted from the left-hand side of a greater than or equal to type constraint to convert the constraint into an equality. It is also known as negative slack variable. In economic terms, surplus variables represent overfulfillment of the requirement.

Specifically:

$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \dots + a_{inx_n} \geq b_i$ can be written as

$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \dots + a_{inx_n} - s_i = b_i$

Where $i = 1, 2, \dots, m$

Artificial variable

It is a non negative variable introduced to facilitate the computation of an initial basic feasible solution. In other words, a variable added to the left-hand side of a greater than or equal to type constraint to convert the constraint into an equality is called an artificial variable.

Simplex Method - Maximization Case

Consider the general linear programming problem

Maximize $z = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n$

subject to

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n \leq b_2$$

.....

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n \leq b_m$$

$$x_1, x_2, \dots, x_n \geq 0$$

Where:

c_j ($j = 1, 2, \dots, n$) in the objective function are called the cost or profit coefficients.

b_i ($i = 1, 2, \dots, m$) are called resources.

a_{ij} ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) are called technological coefficients or input-output coefficients.

Converting inequalities to equalities

Introducing slack variables to convert inequalities to equalities

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n + s_1 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n + s_2 = b_2$$

.....

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n + s_m = b_m$$

$$x_1, x_2, \dots, x_n \geq 0$$

$$s_1, s_2, \dots, s_m \geq 0$$

An initial basic feasible solution is obtained by setting $x_1 = x_2 = \dots = x_n = 0$

$$s_1 = b_1$$

$$s_2 = b_2$$

.....

$$s_m = b_m$$

The initial simplex table is formed by writing out the coefficients and constraints of a LPP in a systematic tabular form. The following table shows the structure of a simplex table.

Structure of a simplex table

	c_j	c_1	c_2	c_3	---	c_n	
c_B	Basic variables B	x_1	x_2	x_3	---	x_n	Solution values $b (=XB)$
c_{B1}	x_1	a_{11}	a_{12}	a_{13}	---	a_{1n}	b_1
c_{B2}	x_2	a_{21}	a_{22}	a_{23}	---	a_{2n}	b_2
c_{B3}	x_3	a_{31}	a_{32}	a_{33}	---	a_{3n}	b_3
---	-----	----	----	-----	---	----	-----
c_{Bm}	x_n	a_{m1}	a_{m2}	a_{m3}	---	a_{mn}	b_m
$z_j - c_j$		$z_1 - c_1$	$z_2 - c_2$	$z_3 - c_3$	---	$z_n - c_n$	

Where:

c_j = coefficients of the variables $(m + n)$ in the objective function.

c_B = coefficients of the current basic variables in the objective function.

B = basic variables in the basis.

XB = solution values of the basic variables.

$z_j - c_j$ = index row.

Two Phase Method

In this method, the whole procedure of solving a linear programming problem involving artificial variables is divided into two phases. In phase I, a new objective function is formed by assigning zero to every original variable (including slack and surplus variables) and -1 to each of the artificial variables. Then the artificial variables are eliminated from the basis. The solution at the end of phase I serves as a basic feasible solution for phase II. In phase II, the original objective function is introduced and the usual simplex algorithm is used to find an optimal solution.

The Big M Method

It is a modified version of the simplex method, in which a very large value (M) is assigned to each of the artificial variables.