OPERATIONS RESEARCH -SIMPLEX METHOD

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SIMPLEX METHOD

For linear programming problems involving two variables, the graphical solution method is convenient. However, for problems involving more than two variables or problems involving a large number of constraints, it is better to use solution methods that are adaptable to computers. One such method is called the simplex method, developed by George Dantzig in 1946. It provides us with a systematic way of examining the vertices of the feasible region to determine the optimal value of the objective function.

SIMPLEX METHOD

Suppose we want to find the maximum value of $z = 4x_1 + 6x_2$, where $x_1 \geq 0$ and $x_2 \geq 0$, subject to the following constraints.

$$\begin{aligned} -x_1 + x_2 &\leq 11 \\ x_1 + x_2 &\leq 27 \\ 2x_1 + 5x_2 &\leq 90 \end{aligned}$$

Since the left-hand side of each *inequality* is less than or equal to the right-hand side, there must exist nonnegative numbers s_1, s_2 and s_3 that can be added to the left side of each equation to produce the following system of linear *equations*.

$$\begin{aligned} -x_1 + x_2 + s_1 &= 11 \\ x_1 + x_2 + s_2 &= 27 \\ 2x_1 + 5x_2 + s_3 &= 90 \end{aligned}$$

The numbers s_1, s_2 and s_3 are called **slack variables** because they take up the "slack" in each inequality.

SIMPLEX METHOD

Standard Form of a Linear Programming Problem

A linear programming problem is in **standard form** if it seeks to *maximize* the objective function $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m \end{aligned}$$

where $x_i \geq 0$ and $b_i \geq 0$. After adding slack variables, the corresponding system of **constraint equations** is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 \qquad\qquad &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad + s_2 \qquad &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad + s_m &= b_m \end{aligned}$$

where $s_i \geq 0$.

SIMPLEX METHOD

A **basic solution** of a linear programming problem in standard form is a solution $(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m)$ of the constraint equations in which *at most m* variables are nonzero—the variables that are nonzero are called **basic variables.** A basic solution for which all variables are nonnegative is called a **basic feasible solution.**

The Simplex Tableau

The simplex method is carried out by performing elementary row operations on a matrix that we call the **simplex tableau.** This tableau consists of the augmented matrix corresponding to the constraint equations together with the coefficients of the objective function written in the form

$$-c_1x_1 - c_2x_2 - \cdots - c_nx_n + (0)s_1 + (0)s_2 + \cdots + (0)s_m + z = 0.$$

In the tableau, it is customary to omit the coefficient of z. For instance, the simplex tableau for the linear programming problem

SIMPLEX METHOD

$$z = 4x_1 + 6x_2$$

Objective function

$$\left.\begin{aligned} -x_1 + x_2 + s_1 &= 11 \\ x_1 + x_2 + s_2 &= 27 \\ 2x_1 + 5x_2 + s_3 &= 90 \end{aligned}\right\}$$

Constraints

is as follows.

x_1	x_2	s_1	s_2	s_3	b	*Basic Variables*
−1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
−4	−6	0	0	0	0	

↑ *Current z–value*

SIMPLEX METHOD

For this **initial simplex tableau,** the **basic variables** are s_1, s_2, and s_3, and the **nonbasic variables** (which have a value of zero) are x_1 and x_2. Hence, from the two columns that are farthest to the right, we see that the current solution is

$$x_1 = 0, \quad x_2 = 0, \quad s_1 = 11, \quad s_2 = 27, \quad \text{and} \quad s_3 = 90.$$

This solution is a basic feasible solution and is often written as

$$(x_1, x_2, s_1, s_2, s_3) = (0, 0, 11, 27, 90).$$

The entry in the lower–right corner of the simplex tableau is the current value of z. Note that the bottom–row entries under x_1 and x_2 are the negatives of the coefficients of x_1 and x_2 in the objective function

$$z = 4x_1 + 6x_2.$$

To perform an **optimality check** for a solution represented by a simplex tableau, we look at the entries in the bottom row of the tableau. If any of these entries are negative (as above), then the current solution is *not* optimal.

SIMPLEX METHOD-ALGORITHM

To solve a linear programming problem in standard form, use the following steps.

1. Convert each inequality in the set of constraints to an equation by adding slack variables.
2. Create the initial simplex tableau.
3. Locate the most negative entry in the bottom row. The column for this entry is called the **entering column.** (If ties occur, any of the tied entries can be used to determine the entering column.)
4. Form the ratios of the entries in the "*b*-column" with their corresponding positive entries in the entering column. The **departing row** corresponds to the smallest nonnegative ratio b_i/a_{ij}. (If all entries in the entering column are 0 or negative, then there is no maximum solution. For ties, choose either entry.) The entry in the departing row and the entering column is called the **pivot.**
5. Use elementary row operations so that the pivot is 1, and all other entries in the entering column are 0. This process is called **pivoting.**
6. If all entries in the bottom row are zero or positive, this is the final tableau. If not, go back to Step 3.
7. If you obtain a final tableau, then the linear programming problem has a maximum solution, which is given by the entry in the lower-right corner of the tableau.

SIMPLEX METHOD

Note that the basic feasible solution of an initial simplex tableau is

$$(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m) = (0, 0, \ldots, 0, b_1, b_2, \ldots, b_m).$$

This solution is basic because at most m variables are nonzero (namely the slack variables). It is feasible because each variable is nonnegative.

In the next two examples, we illustrate the use of the simplex method to solve a problem involving three decision variables.

The Simplex Method with Three Decision Variables

Use the simplex method to find the maximum value of

$$z = 2x_1 - x_2 + 2x_3 \qquad \text{Objective function}$$

subject to the constraints

$$\begin{aligned} 2x_1 + x_2 \phantom{{}- 2x_3} &\leq 10 \\ x_1 + 2x_2 - 2x_3 &\leq 20 \\ x_2 + 2x_3 &\leq 5 \end{aligned}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$.

SIMPLEX METHOD

Solution Using the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 10, 20, 5)$$

the initial simplex tableau for this problem is as follows.

x_1	x_2	x_3	s_1	s_2	s_3	b	*Basic Variables*	
2	1	0	1	0	0	10	s_1	
1	2	−2	0	1	0	20	s_2	
0	1	(2)	0	0	1	5	s_3	← *Departing*
−2	1	−2	0	0	0	0		
		↑ *Entering*						

SIMPLEX METHOD

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
2	1	0	1	0	0	10	s_1	← *Departing*
1	3	0	0	1	1	25	s_2	
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3	
−2	2	0	0	0	1	5		
↑ *Entering*								

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	5	x_1
0	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	1	20	s_2
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3
0	3	0	1	0	1	15	

SIMPLEX METHOD

This implies that the optimal solution is

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (5, 0, \tfrac{5}{2}, 0, 20, 0)$$

and the maximum value of z is 15.

THANK YOU